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Temperature effects on a network of dissipative quantum harmonic oscillators: collective damping and dispersion processes

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Abstract

In this paper we extend the results presented in (de Ponte, Mizrahi and Moussa 2007 *Phys. Rev.* A **76** 032101) to treat quantitatively the effects of reservoirs at finite temperature in a bosonic dissipative network: a chain of coupled harmonic oscillators whatever its topology, i.e., whichever the *way* the oscillators are coupled together, the *strength* of their couplings and their *natural frequencies*. Starting with the case where distinct reservoirs are considered, each one coupled to a corresponding oscillator, we also analyze the case where a common reservoir is assigned to the whole network. Master equations are derived for both situations and both regimes of weak and strong coupling strengths between the network oscillators. Solutions of these master equations are presented through the normal ordered characteristic function. These solutions are shown to be significantly involved when temperature effects are considered, making difficult the analysis of collective decoherence and dispersion in dissipative bosonic networks. To circumvent these difficulties, we turn to the Wigner distribution function which enables us to present a technique to estimate the decoherence time of network states. Our technique proceeds by computing separately the effects of dispersion and the attenuation of the interference terms of the Wigner function. A detailed analysis of the dispersion mechanism is also presented through the evolution of the Wigner function. The interesting collective dispersion effects are discussed and applied to the analysis of decoherence of a class of network states. Finally, the entropy and the entanglement of a pure bipartite system are discussed.

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1. Introduction

The subject of networks of interacting quantum systems has acquired an important role in the emerging field of quantum information theory. Since a realistic quantum logic processor must ultimately be composed of a large number of interacting quantum systems, it becomes mandatory to understand processes such as perfect state transfer from one system to another system of the network, and even to compute the fidelity of such a state transfer when the action of the environment is taken into account. A significant number of results have recently been derived on the subject of perfect state transfer in optical lattices [1], networks of spin [2] and harmonic oscillators [3]. Perfect state transfer has also been considered in networks of arbitrary topology and coupling configuration [4] and even under random fluctuations in the couplings of a quantum chain [5].

Apart from state transfer, the process of decoherence of a network state has also attracted attention and interesting properties of collective damping effects, such as the nonadditivity of decoherence rates have been discussed in different contexts as in superconducting qubits [6], two-atom systems [7] and chains of dissipative harmonic oscillators [8–12]. Still regarding the process of collective decoherence, the emergence of decoherence-free subspaces (DFSs) has also instigated several interesting results when considering the particular case of a composite system interacting with a common reservoir [13], or the more realistic situation where each system interacts with its own reservoir [10]. We call attention to the fact that all of [8–10] envisage such realistic cases of networks where each oscillator interacts with its own reservoir, and also address the particular case where a common reservoir is considered. Therefore, beyond the analysis of decoherence due to the inevitable coupling of a single quantum system with a reservoir, as in [14], in this paper we are concerned with the collective mechanisms of decoherence and dispersion in a thermal dissipative bosonic network.

To better understand the results in [8–10], which are crucial to introducing the subject of the present work, we remember that, apart from the distinct reservoirs, the network of N dissipative harmonic oscillators could present direct and indirect dissipative channels. Through the direct channels each oscillator loses excitation to its own reservoir, whereas through the indirect channels it loses excitation to all the other reservoirs but not to its own. When we consider distinct reservoirs for each network oscillator, the indirect dissipative channels—intrinsically associated with the nonadditivity of decoherence rates and the emergence of DFSs [10]—are significant only in the strong coupling regime where $N\lambda_{mn} \simeq \omega_\ell$, i.e., the number of network oscillators N multiplied by their coupling strengths $\{\lambda_{mn}\}$ is about their natural frequencies $\{\omega_m\}$. Therefore, the strong coupling regime, which brings together the collective damping effects, depends on the number of network oscillators as much as on their coupling strengths. For Markovian white noise reservoirs, however, where the spectral densities of the reservoirs are invariant under translations in frequency space, the indirect channels become null, except for the case of $N = 2$ [8].

In the weak coupling regime where $N\lambda_{mn} \ll \omega_\ell$, the indirect channels always disappear. However, these indirect channels, coming from the strong coupling regime, remain in the case where all network oscillators interact with a common reservoir [10], even assuming a common Markovian white noise reservoir. This is due to the fact that a common reservoir induces an additional correlation between the network oscillators, restoring the indirect decay channels.

Recently, a generalization of [8, 9] has been presented through a comprehensive treatment of networks of dissipative quantum harmonic oscillators, whatever its topology, i.e., whichever the *way* the oscillators are coupled together, the *strength* of their couplings and their *natural frequencies* [11]. Focusing on the general more realistic scenario where each oscillator is coupled to its own reservoir, the case where all the network oscillators are coupled to

a common reservoir was also addressed. However, after deducing the master equation for the case where all the reservoirs are at finite temperatures, all further analysis of the dynamics of the network states was restricted to the case where the reservoirs are at $T = 0$ K. Whereas a quantitative analysis of the decoherence and the evolution of the linear entropy of representative states of the network were given at 0 K, only a brief qualitative analysis of the equilibrium states of the network was presented at finite temperatures. In the present manuscript we extend the treatment in [11] given a detailed analysis of the temperature effects on networks of dissipative quantum harmonic oscillators.

The present extension of [11] that accounts for the temperature effects coming from thermal reservoirs is not only interesting due to its more realistic approach but also from the mathematical development here achieved. In fact, we present an alternative approach to previous results in the literature [15] regarding the obtainment of the solution of the master equation and the estimation of decoherence times through the Wigner distribution function. As a matter of fact, since the solutions of our master equations (for distinct reservoirs or a common one, under the weak and strong coupling regimes) appear to be significantly involved, we turn to the Wigner distribution function to analyze collective decoherence and dispersion in our dissipative networks. Being easier to handle mathematically, the Wigner function enabled us to present a technique to estimate the decoherence time of network states. We observe that the time-dependent dispersion mechanism, to be defined below, follows directly from the time-independent diffusion coefficients taking place in the master equations associated with Markovian reservoirs. Evidently, when the diffusion coefficients are null, for reservoirs at $T = 0$ K, the time-dependent dispersion process disappears, as in [16].

To circumvent noise effects, many of the nowadays experiments demonstrating quantum logic operations through atom–field interactions occur in cryogenic environments where, in general, temperature effects are negligible. In cavity quantum electrodynamics, the setup is cooled to around 0.5 K by a ^{3}He–^{4}He refrigerator to avoid blackbody radiation in the high-Q superconducting cavity. Under such a specific condition, the temperature effects on the decoherence process are almost negligible. However, when the setup is scaled from one single cavity to a network of N high-Q cavities, major questions arise due to temperature effects. First of all, would the DFSs survive despite the temperature effects? Apart from the special class of states composing the DFSs, how the temperature affects other states of the network, as for example initial entangled states? Evidently, these questions present no obvious answers, even under the assumption that all network cavities are cooled at low temperatures. In this regard, we expect the collective damping effects coming from the indirect dissipative channels to play a major role for the answers to the above questions. It is important to emphasize that some systems undergo a significant degree of decoherence even for cryogenic temperatures, as for example a particle in a Penning trap with harmonic confinement, where the typical frequencies ω do not obey the relation $\hbar\omega \gg k_B T$, with k_B being the Boltzmann constant [17]. In this regard we must underline the relevance of the present analysis for recently proposed schemes for quantum computation based on an array of Penning traps [18].

Apart from providing the mathematical treatment of the temperature effects on a network of N dissipative harmonic oscillators, in the present manuscript we also analyze the role played by temperature on the evolution of particular states of the network other than those composing DFSs. We reserve the analyses of the emergence of DFSs under temperature effects to a specific work [19] where the mechanism of construction of such privileged states is also discussed along with decoherence.

This paper is organized as follows: in section 2 we revisit our model of a bosonic dissipative network [11] and present the derivation of the master equation governing the dynamics of the associated density operator. In section 3 we present the solution of the

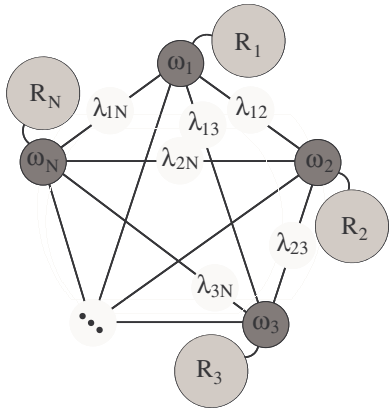

Figure 1. Sketch of a dissipative symmetric network of N oscillators, where each one interacts with each other, apart from its own reservoir.

normal ordered characteristic equation obtained from the master equation for the density operator of the network. In section 4 we analyze the evolution of two general classes of initial states of the network, given by mixtures of coherent and number states, through the normal ordered characteristic equation, the Glauber–Sudarshan P-function, and the Wigner distribution. A detailed analysis of the dispersion processes is presented in section 5 and the collective decoherence rates of a family of states of the network are analyzed in section 6. In section 7 we discuss the entropy and the entanglement degree of a pure bipartite system and, finally, in section 8 we present our concluding remarks.

2. The master equation of a bosonic dissipative network

We present here a brief review of the steps for the derivation of the master equation of a bosonic network, as developed in [11]. We start from the general case of a network of N interacting oscillators, where each one interacts with each other, from which all other topologies can be recovered. As depicted in figure 1, we also consider the case where each oscillator interacts with its own reservoir due to this being a more realistic approach for most of the physical systems. However, as pointed in [10], despite the realistic scenario of the case of distinct reservoirs, the case of a common reservoir is more general from the technical point of view. In fact, as discussed at the end of this section, the master equation for the case of distinct reservoirs can be deduced from the case of a common reservoir.

We start by considering a general Hamiltonian for a bosonic network, $H = H_S + H_R + H_I$, composed by a set of N coupled oscillators

$$H_S = \hbar \sum_{m=1}^{N} \left[\omega_m a_m^\dagger a_m + \frac{1}{2} \sum_{n(\neq m)=1}^{N} \lambda_{mn} \left(a_m^\dagger a_n + a_m a_n^\dagger \right) \right], \tag{1}$$

N distinct reservoirs, modeled by a set of $k = 1, \ldots, \infty$ modes,

$$H_R = \hbar \sum_{m=1}^{N} \sum_{k} \omega_{mk} b_{mk}^\dagger b_{mk}, \tag{2}$$

and the coupling between the network oscillators and their respective reservoirs

$$H_I = \hbar \sum_{m=1}^{N} \sum_{k} V_{mk} \left(b_{mk}^{\dagger} a_m + b_{mk} a_m^{\dagger} \right), \tag{3}$$

where $b_{mk}^{\dagger}$ (b_{mk}) is the creation (annihilation) operator for the kth bath mode ω_{mk} coupled to the mth network oscillator ω_m whose creation (annihilation) operator reads $a_m^{\dagger}$ (a_m). The coupling strengths between the oscillators are given by the set $\{\lambda_{mn}\}$, while those between the oscillators and their reservoirs by $\{V_{mk}\}$. We assume, from here on, that ℓ, m, n, ℓ', m' and n' run from 1 to N.

Before addressing the dissipative process through Hamiltonian (3), we focus first on Hamiltonian H_S to show how to derive different topologies of a nondissipative network of coupled harmonic oscillators. Rewriting H_S in a matrix form

$$H_S = \hbar \begin{pmatrix} a_1^{\dagger} & \cdots & a_N^{\dagger} \end{pmatrix} \begin{pmatrix} \mathcal{H}_{11} & \cdots & \mathcal{H}_{1N} \\ \vdots & \ddots & \vdots \\ \mathcal{H}_{N1} & \cdots & \mathcal{H}_{NN} \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix}, \tag{4}$$

we identify the elements of the matrix $\mathcal{H} = \mathcal{H}^{\dagger}$ as

$$\mathcal{H}_{mn} = \begin{cases} \omega_m & \text{for} \quad m = n \\ \lambda_{mn} & \text{for} \quad m \neq n \end{cases}, \tag{5}$$

whose values characterize whatever the network topology, i.e., whichever the way the oscillators are coupled together, the set of coupling strengths $\{\lambda_{mn}\}$, and their natural frequencies $\{\omega_m\}$.

To obtain the master equation of the network we first diagonalize the Hamiltonian $\mathcal{H}$ (within the physical regime where the normal modes are assumed to be positive) through the canonical transformation

$$A_m = \sum_{n} C_{mn} a_n, \tag{6}$$

where the coefficients of the mth line of matrix $\mathbf{C}$ define the eigenvectors associated with the eigenvalues ϖ_m of matrix $\mathcal{H}$. With $\mathbf{C}$ being an orthogonal matrix, its transposed $\mathbf{C}^{\intercal}$ turns out to be exactly its inverse $\mathbf{C}^{-1}$, resulting in the commutation relations $\left[A_m, A_n^{\dagger}\right] = \delta_{mn}$ and $[A_m, A_n] = 0$, which enable the Hamiltonian H to be rewritten as a sum $H = H_0 + V$, where

$$H_0 = \hbar \sum_{m} \varpi_m A_m^{\dagger} A_m + \hbar \sum_{m} \sum_{k} \omega_{mk} b_{mk}^{\dagger} b_{mk}, \tag{7a}$$

$$V = \hbar \sum_{m,n} \sum_{k} C_{mn}^{-1} V_{mk} \left(b_{mk}^{\dagger} A_n + b_{mk} A_n^{\dagger} \right). \tag{7b}$$

With the diagonalized Hamiltonian H_0 we are ready to introduce the interaction picture, defined by the transformation $U_0(t) = \exp\left(-\mathrm{i} H_0 t/\hbar\right)$, where

$$V(t) = \hbar \sum_{m,n} \left[\mathcal{O}_{mn}(t) A_n^{\dagger} + \mathcal{O}_{mn}^{\dagger}(t) A_n \right], \tag{8}$$

and $\mathcal{O}_{mn}(t) = C_{mn}^{-1} \sum_k V_{mk} \exp\left[-\mathrm{i}\left(\omega_{mk} - \varpi_n\right) t\right] b_{mk}$. Next, we assume that the interactions between the resonators and the reservoirs are weak enough to allow a second-order perturbation approximation. We also assume a Markovian reservoir such that the density operator of the

global system can be factorized as $\rho_S(t)\otimes\rho_R(0)$. Under these assumptions the reduced density operator of the network of N dissipative coupled resonators satisfy the differential equation

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -\frac{1}{\hbar^2}\int_0^t \mathrm{d}\tau\, \mathrm{Tr}_R\left[V(t),\left[V(\tau),\rho_S(t)\otimes\rho_R(0)\right]\right]. \tag{9}$$

Since for a thermal reservoir $\langle b_{mk}b_{nk'}\rangle = \left\langle b^\dagger_{mk}b^\dagger_{nk'}\right\rangle = 0$, we have to solve the integrals appearing in equation (9), related to correlation functions of the form

$$\int_0^t \mathrm{d}\tau\left\langle \mathcal{O}_{mn}(t)\mathcal{O}^\dagger_{m'\ell}(\tau)\right\rangle = C^{-1}_{mn}C_{\ell m'}\int_0^t \mathrm{d}\tau\sum_{k,k'} V_{mk}V_{m'k'}\left\langle b_{mk}b^\dagger_{m'k'}\right\rangle \times \exp\{-\mathrm{i}[(\omega_{mk}-\varpi_n)t-(\omega_{m'k'}-\varpi_\ell)\tau]\}. \tag{10}$$

Considering that the reservoir frequencies are very closely spaced to allow a continuum summation, we obtain

$$\int_0^t \mathrm{d}\tau\left\langle \mathcal{O}_{mn}(t)\mathcal{O}^\dagger_{m'\ell}(\tau)\right\rangle = N\delta_{mm'}C^{-1}_{mn}C_{\ell m}\frac{\gamma_m(\varpi_\ell)+\bar{n}_m(\varpi_\ell)\tilde{\gamma}_m(\varpi_\ell)}{2}\,\mathrm{e}^{i(\varpi_\ell-\varpi_n)t}, \tag{11}$$

where we have defined the average excitation of the reservoir associated with the mth oscillator as $\bar{n}_m(\nu)$ through the relation $\left\langle b^\dagger_m(\nu)b_n(\nu')\right\rangle = 2\pi\delta_{mn}\bar{n}_m(\nu)\delta(\nu-\nu')$, apart from the damping rates

$$\gamma_m(\omega) = \int_0^t \mathrm{d}\tau\int_0^\infty \frac{\mathrm{d}\nu}{N\pi}\left[V_m(\nu)\sigma_m(\nu)\right]^2\,\mathrm{e}^{-\mathrm{i}(\nu-\omega)(\tau-t)}, \tag{12a}$$

$$\tilde{\gamma}_m(\omega) = \int_0^t \mathrm{d}\tau\int_0^\infty \frac{\mathrm{d}\nu}{N\pi}\left[V_m(\nu)\sigma_m(\nu)\right]^2\frac{\bar{n}_m(\nu)}{\bar{n}_m(\omega)}\,\mathrm{e}^{-\mathrm{i}(\nu-\omega)(\tau-t)}, \tag{12b}$$

with $\sigma_m(\nu)$ being the density of states of the mth reservoir. In the context of Markov approximation, where $V_m(\varpi_n)$, $\sigma_m(\varpi_n)$ and $\bar{n}_m(\varpi_n)$ are slowly varying functions around the normal modes ϖ_n we can simplify the expressions (12) to their usual forms

$$\gamma_m(\omega) = \tilde{\gamma}_m(\omega) = \frac{1}{N}\left[V_m(\omega)\sigma_m(\omega)\right]^2. \tag{13}$$

Back to the Schrödinger picture and to the original field operators a_m, we finally obtain from the steps outlined above, the master equation

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = \frac{i}{\hbar}\left[\rho_S(t),H_S\right] + \sum_{m,n}\left[\frac{\Gamma_{mn}+\Upsilon_{mn}}{2}\mathcal{L}_{mn}\rho_S(t) + \frac{\Upsilon_{mn}}{2}\mathfrak{L}_{mn}\rho_S(t)\right], \tag{14}$$

where we have defined the damping and the diffusion matrix elements Γ_{mn} and Υ_{mn} in the forms

$$\Gamma_{mn} = N\sum_\ell C_{\ell n}\gamma_m(\varpi_\ell)C^{-1}_{m\ell}, \tag{15a}$$

$$\Upsilon_{mn} = N\sum_\ell C_{\ell n}\tilde{\gamma}_m(\varpi_\ell)\bar{n}_m(\varpi_\ell)C^{-1}_{m\ell}, \tag{15b}$$

whereas the Liouville operators accounting for the *direct* ($m = n$) and *indirect* ($m \neq n$) dissipative channels are given by

$$\mathcal{L}_{mn}\rho_S(t) \equiv \left[a_n\rho_S(t),a^\dagger_m\right] + \left[a_m,\rho_S(t)a^\dagger_n\right], \tag{16a}$$

$$\mathfrak{L}_{mn}\rho_S(t) \equiv \left[a^\dagger_n\rho_S(t),a_m\right] + \left[a^\dagger_m,\rho_S(t)a_n\right]. \tag{16b}$$

We note that, as expected for Markovian reservoirs [20], the diffusion matrix elements are time independent, differing from those arising from non-Markovian reservoirs [21].

As mentioned in section 1 and discussed in [11], the oscillators lose excitation to their own reservoirs through the direct dissipative channels, whereas through the indirect channels they lose excitation to all the other reservoirs but not to their own. Although in [11] we have obtained the master equation for the general case of reservoirs at finite temperatures, all further analysis was carried out for reservoirs at 0 K where the diffusion matrix elements Υ_{mn} are null. Next, considering the case of reservoirs at finite temperatures, we must discuss the master equation (14) under the weak and strong coupling regimes between the network oscillators. Here we must emphasize that while we assume the interactions between the resonators and the reservoirs to be weak, those between the networks oscillators can also be strong, in the sense to be defined below.

2.1. Weak interoscillator coupling regime

We first remember that the weak coupling regime for a network of N coupled oscillators is defined by the relation $N\lambda_{mn} \ll \omega_\ell$. (However, if a specific coupling λ_{mn} between two oscillators, m and n, fails to satisfy the relation $N\lambda_{mn} \ll \omega_\ell$, the network dynamics is necessarily described by the strong coupling regime, with some normal-mode frequencies far beyond their natural values.) In the weak coupling regime, the interaction between the network oscillators, described by $\hbar \sum_{m,\neq n} \lambda_{mn}\left(a_m^\dagger a_n + a_m a_n^\dagger\right)/2$, could be directly introduced into the von Neumann term of the master equation to a good approximation, circumventing the necessity to diagonalize the Hamiltonian $\mathcal{H}$ through a canonical transformation $A_m = \sum_n C_{mn} a_n$. This is equivalent to approximating the matrix $\mathbf{C}$ by a identity matrix $\mathbf{I}$, implying that

$$\Gamma_{mn} = N\gamma_m(\omega_m)\delta_{mn}, \tag{17a}$$

$$\Upsilon_{mn} = N\tilde{\gamma}_m(\omega_m)\bar{n}_m(\omega_m)\delta_{mn}, \tag{17b}$$

where, evidently, we have also approximated the normal modes by their original natural frequencies. Under the above considerations, the master equation (14) becomes

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = \frac{i}{\hbar}\left[\rho_S(t), H_S\right] + N\sum_m \left[\frac{\gamma_m(\omega_m) + \tilde{\gamma}_m(\omega_m)\bar{n}_m(\omega_m)}{2}\mathcal{L}_{mm}\rho_S(t) + \frac{\tilde{\gamma}_m(\omega_m)\bar{n}_m(\omega_m)}{2}\mathfrak{L}_{mm}\rho_S(t)\right], \tag{18}$$

where, essentially, the indirect dissipative channels disappear, establishing the additivity of the decoherence rates.

2.2. Strong interoscillator coupling regime

The strong coupling regime means that $N\lambda_{mn} \approx \omega_\ell$, i.e., at least one of the couplings $\{\lambda_{mn}\}$ between two network oscillators must be of the order of any natural frequency $\{\omega_m\}$. In this case, the indirect dissipative channels become effective, inducing collective damping and dispersion effects that we must investigate. As pondered in section 1, would the collective effects of nonadditivity of the decay rates and the emergence of DFSs still survive despite the temperature effects?

It must be mentioned that Markovian white noise reservoirs wash out the collective damping effects introduced by the strong coupling regime since the spectral densities are

invariant under translations in frequency space, i.e., $\gamma_m(\varpi_n) = \gamma_m$, rendering the same matrix elements Γ_{mn} as in equation (17*a*). However, Markovian white noise reservoirs do not wash out the collective diffusion effects. In fact, only under the *additional* assumption that $\bar{n}_m(\varpi_n) \approx \bar{n}_m$ for whatever normal mode ϖ_n, we recover equation (17*b*) for the diffusion matrix $\boldsymbol{\Upsilon}$, erasing the collective effects completely. Next we discuss the case where the whole network is under the action of a common reservoir.

2.3. A common reservoir

When all the network oscillators are coupled to a single reservoir, the master equation, derived in [10], is similar to that in equation (14), replacing the damping and the diffusion matrix elements by

$$\Gamma_{mn} = N \sum_{\ell,n'} C_{\ell n} \gamma_{mn'}(\varpi_\ell)\, C^{-1}_{n'\ell}, \tag{19a}$$

$$\Upsilon_{mn} = N \sum_{\ell,n'} C_{\ell n} \tilde{\gamma}_{mn'}(\varpi_\ell)\, C^{-1}_{n'\ell}, \tag{19b}$$

where the damping rates $\gamma_{mn}(\omega)$ and $\tilde{\gamma}_{mn}(\omega)$ for the case of a single common reservoir are given by [10]

$$\gamma_{mn}(\omega) = \int_0^t \mathrm{d}\tau \int_0^\infty \frac{\mathrm{d}\nu}{N\pi} V_m(\nu) V_n(\nu) \sigma^2(\nu)\, \mathrm{e}^{-\mathrm{i}(\nu-\omega)\tau}, \tag{20a}$$

$$\tilde{\gamma}_{mn}(\omega) = \int_0^t \mathrm{d}\tau \int_0^\infty \frac{\mathrm{d}\nu}{N\pi} V_m(\nu) V_n(\nu) \sigma^2(\nu) \frac{\bar{n}(\nu)}{\bar{n}(\omega)}\, \mathrm{e}^{-\mathrm{i}(\nu-\omega)\tau}. \tag{20b}$$

As mentioned above and discussed in [10], the master equation for the case of distinct reservoirs can be deduced from the case of a single common reservoir. The above deduction of the master equation (14), where we started from the case of distinct reservoirs, was entirely due to its broad application in many physical systems. To demonstrate how to derive the case of a distinct reservoir from that of a common one, we remember that $V_m(\nu)$ gives the distribution function of the reservoir modes coupled to the mth oscillator. Therefore, in the absence of overlap between the distribution functions, i.e., $\int \mathrm{d}\nu V_m(\nu) V_n(\nu) = 0$ for $m \neq n$, equations (20) reduce to those in equations (12). In this case, the occurrence of the indirect-decay channels follows entirely from the strong coupling between the oscillators, as discussed in the subsections presented above. When there is a significant overlap between the distribution functions, i.e., $\int \mathrm{d}\nu V_m(\nu) V_n(\nu) \neq 0$ for at least one $m \neq n$, we get the indirect-decay channels even when the network oscillators do not interact at all. The strength of the damping and the diffusion matrix elements being defined by the amount of the overlap, i.e., when the overlap between the distributions $V_m(\nu)$ and $V_n(\nu)$ is maximum, the strengths Γ_{mn} and Υ_{mn} equal Γ_{mm} and Υ_{mm}.

3. Normal ordered characteristic function

To analyze the dynamics of the network states for the case where the reservoirs are at finite temperatures, we consider the evolution of the (normal ordered) characteristic function, derived from the master equation (14) (suitable for all cases discussed in the previous section) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\chi(\{\eta_m\}, t) = -\sum_{m,n}\left[\eta_m \frac{\Upsilon_{mn}}{2}\eta_n^* + \eta_m \left(\mathcal{H}^D_{mn}\right)^* \frac{\partial}{\partial \eta_n} + c.c.\right]\chi(\{\eta_m\}, t), \tag{21}$$

where we defined the matrix elements

$$\mathcal{H}^D_{mn} = \Gamma_{mn}/2 + \mathrm{i}\mathcal{H}_{mn}. \tag{22}$$

As noted in [11], the matrix $\mathcal{H}^D$ is an extension of the free evolution $\mathcal{H}$ in equation (5), which takes into account the dissipative mechanisms of the network.

Starting with the assumption that equation (21) admits a solution of the form $\chi(\{\eta_m\}, t) = \varphi(\{\eta_m\})\phi(\{\eta_m\}, t)$, we obtain two differential equations, one accounting for the dynamic process, given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(\{\eta_m\}, t) = -\sum_{m,n}\left[\eta_m \left(\mathcal{H}^D_{mn}\right)^* \frac{\partial}{\partial \eta_n} + \text{c.c.}\right]\phi(\{\eta_m\}, t), \tag{23}$$

and the other accounting for the stationary solution of the characteristic function, given by

$$\sum_{m,n}\left[\eta_m \frac{\Upsilon_{mn}}{2}\eta^*_n + \eta_m \left(\mathcal{H}^D_{mn}\right)^* \frac{\partial}{\partial \eta_n} + \text{c.c.}\right]\varphi(\{\eta_m\}) = 0. \tag{24}$$

If we perform the substitution $\mathcal{H}^D \to -(\mathcal{H}^D)^\dagger$ into the first differential equation (23), it turns out to be exactly that appearing in [11] for the derivation of the solution of the Glauber–Sudarshan P-function. Therefore, following the steps outlined in [11], the solution of equation (23) can be written as

$$\eta_m(t) = \sum_{\ell,n} \eta_n(0) D^*_{n\ell} \exp(-\Omega^*_\ell t)\left(D^{-1}_{\ell m}\right)^*, \tag{25}$$

where we employed the diagonal form of $\mathcal{H}^D$ following from the transformation $\mathbf{D}^{-1} \bullet \mathcal{H}^D \bullet \mathbf{D} = \mathbf{\Omega}$. Note that writing solution (25) in a matrix form, it becomes

$$\begin{aligned}\boldsymbol{\eta}(t) &= \boldsymbol{\eta}(0) \bullet \mathbf{D}^* \bullet \exp(-\mathbf{\Omega}^* t) \bullet (\mathbf{D}^{-1})^* \\ &= \boldsymbol{\eta}(0) \bullet \exp[-(\mathbf{D} \bullet \Omega \bullet \mathbf{D}^{-1})^* t] \\ &= \boldsymbol{\eta}(0) \bullet \exp[-(\mathcal{H}^D)^* t]\end{aligned} \tag{26}$$

such that

$$\frac{\mathrm{d}\boldsymbol{\eta}(t)}{\mathrm{d}t} = -\boldsymbol{\eta}(t) \bullet (\mathcal{H}^D)^*, \tag{27}$$

or, equivalently,

$$\frac{\mathrm{d}\eta_n(t)}{\mathrm{d}t} = -\sum_m \eta_m(t)\left(\mathcal{H}^D_{mn}\right)^*, \tag{28}$$

representing a system of coupled differential equations which follows from equation (23) under the assumption that $\phi(\{\eta_m\}, t) = \phi(\{\eta_m(t)\})$, with $\eta_m = \eta_m(0)$ [11].

The second differential equation (24) can be solved assuming a general Gaussian form

$$\varphi(\{\eta_m\}) = \exp\left(-\frac{1}{2}\sum_{m,n}\eta_m \Pi_{mn}\eta^*_n\right), \tag{29}$$

where the elements of matrix $\mathbf{\Pi}$ are the coefficients to be determined. Substituting (29) into equation (24) and changing conveniently the labels m and n of the involved matrices, we verify that the differential equation (24) reduces to a matrix equation of the form

$$(\mathcal{H}^D)^* \bullet \mathbf{\Pi} + \Pi \bullet (\mathcal{H}^D)^\top = \mathbf{\Upsilon} + \mathbf{\Upsilon}^\top, \tag{30}$$

which is explicitly written as

$$\sum_\ell \left(\mathcal{H}^D_{m\ell}\right)^* \Pi_{\ell n} + \sum_\ell \Pi_{m\ell}\mathcal{H}^D_{n\ell} = \Upsilon_{mn} + \Upsilon_{nm}. \tag{31}$$

with the superscript $\top$ in equation (30) standing for transposed. It is worth noting that for identical reservoirs, where $\gamma_m = \gamma$ and so $\mathbf{\Gamma} = \mathbf{\Gamma}^\top$, we obtain a symmetric dissipative matrix $\mathcal{H}^D$, i.e., $\mathcal{H}^D = \left(\mathcal{H}^D\right)^\top$, making equation (30) the well-known Lyapunov equation. The solution of equation (30), namely the determination of $\mathbf{\Pi}$, can be obtained by converting the matrix equation into a system of N^2 algebraic equations, i.e., into a new matrix equation of the simplified form $\mathbf{A} \bullet \mathbf{X} = \mathbf{B}$, with the elements of matrix $\mathbf{X}$ being the N^2 unknown variables. To this end, it is useful to define the column vector

$$\mathrm{vec}(\Pi) \equiv (\Pi_{11} \quad \Pi_{21} \quad \cdots \quad \Pi_{N1} \quad \Pi_{12} \quad \cdots \quad \Pi_{N2} \quad \cdots \quad \Pi_{1N} \quad \cdots \quad \Pi_{NN})^\top, \tag{32}$$

where the first N elements of vec (Π) correspond to the first column of matrix $\mathbf{\Pi}$, whereas the next N elements correspond to the second column of $\mathbf{\Pi}$ and so on. As so, the matrix equation (30) can be rewritten into the form [22]

$$[\mathbf{I} \otimes (\mathcal{H}^D)^* + \mathcal{H}^D \otimes \mathbf{I}] \bullet \mathrm{vec}\,(\Pi) = \mathrm{vec}(\mathbf{\Upsilon} + \mathbf{\Upsilon}^\top), \tag{33}$$

where $\mathbf{I}$ is an $N \times N$ identity matrix. From the mathematical properties presented in appendix A for the matrix $\left[\mathbf{I} \otimes \left(\mathcal{H}^D\right)^* + \mathcal{H}^D \otimes \mathbf{I}\right]$, we verify that the elements of matrix $\mathbf{\Pi}$ can be written as

$$\Pi_{\ell\ell'} = \sum_{m,n,m',n'} \frac{\Upsilon_{m'n'} + \Upsilon_{n'm'}}{\Omega_m + \Omega_n^*} D_{\ell' m} D^{-1}_{mm'} \left(D_{\ell n} D^{-1}_{nn'}\right)^*, \tag{34}$$

finally leading to the solution of equation (24) through equation (29). In fact, substituting the expression (34) into the left-hand side of equation (31), and using the relation $\mathbf{D}^{-1} \bullet \mathcal{H}^D \bullet \mathbf{D} = \Omega \Rightarrow \mathcal{H}^D \bullet \mathbf{D} = \mathbf{D} \bullet \Omega$, we obtain

$$\sum_\ell \left[\left(\mathcal{H}^D_{m\ell}\right)^* \Pi_{\ell n} + \Pi_{m\ell} \mathcal{H}^D_{n\ell}\right] = \sum_{m',n'} \left(\Upsilon_{m'n'} + \Upsilon_{n'm'}\right) \delta_{mn'} \delta_{nm'} = \Upsilon_{mn} + \Upsilon_{nm}, \tag{35}$$

which is exactly the right-hand side of equation (31).

We thus verify that the solution of the characteristic equation is of the form

$$\chi(\{\eta_m\}, t) = \varphi(\{\eta_m\})\, [\phi(\{\eta_m\}, t = 0)|_{\{\eta_m\} \Rightarrow \{\eta_m(t)\}}\,. \tag{36}$$

Since for $t = 0$ we get $\chi(\{\eta_m\}, 0) = \varphi(\{\eta_m\})\phi(\{\eta_m\}, 0)$, such that $\phi(\{\eta_m\}, 0) = \chi(\{\eta_m\}, 0)/\varphi(\{\eta_m\})$, we end up with the solution of the characteristic function

$$\chi(\{\eta_m\}, t) = \frac{\varphi(\{\eta_m\})}{\varphi(\{\eta_m(t)\})}\, [\chi(\{\eta_m\}, t = 0)|_{\{\eta_m\} \Rightarrow \{\eta_m(t)\}}\,, \tag{37}$$

given in terms of its initial state. An interesting point to be noted is that the dynamics of the problem, given by $\eta_m(t)$, takes into account only the dissipative rates Γ_{mn} together with the free evolution Hamiltonian $\mathcal{H}$ in equation (5), leaving aside the diffusive process associated with Υ_{mn}. Such a diffusive process, appearing in the ratio $\varphi(\{\eta_m\})/\varphi(\{\eta_m(t)\})$ is, however, modified by the dissipative mechanisms.

4. Dynamics of the network states: characteristic function, Glauber–Sudarshan *P*-function, Wigner distribution and density operator

Starting from two general classes of initial network states, given by mixed superpositions of coherent and number states, we next analyze the evolution of such states through the characteristic function, the Glauber–Sudarshan P-function and the Wigner distribution. We also compute the network density operator for the case of mixed superposition of Fock states.

4.1. A mixed superposition of coherent states

Considering that the initial network state comprehends a mixture of superpositions of coherent states such as $|\Psi(0)\rangle_J = \int \mathrm{d}r_J \Lambda(r_J)|\{\beta_m(r_J)\}\rangle$, the initial density operator becomes

$$\rho_S(0) = \sum_J p_J \int \mathrm{d}r_J \Lambda(r_J) \int \mathrm{d}s_J \Lambda^*(s_J)|\{\beta_m(r_J)\}\rangle\langle\{\beta_m(s_J)\}|, \tag{38}$$

where p_J is the probability associated with the state $|\Psi(0)\rangle_J$. The parameters r_J (s_J) represent a set of variables defining the probability density function $\Lambda(r_J)$, and $|\{\beta_m(r_J)\}\rangle = \bigotimes_m |\beta_m(r_J)\rangle$ stands for a product of coherent states, where$|\beta_m(r_J)\rangle$ represents the state associated with the mth network oscillator. In the particular case where $\Lambda(r_J) = \sum_k \Lambda_k \delta\big(r_J - r_J^{(k)}\big)$, the pure state $|\Psi(0)\rangle_J$ becomes the discrete superposition

$$|\Psi(0)\rangle_J = \sum_k \Lambda_k \left|\left\{\beta_m^k\right\}\right\rangle_J, \tag{39}$$

where we have defined $\left|\left\{\beta_m\left(r_J^{(k)}\right)\right\}\right\rangle \equiv \left|\left\{\beta_m^k\right\}\right\rangle_J$. Through the definition of the time-dependent vector elements $K_m(r_J;t) = \sum_n \Theta_{mn}(t)\beta_n(r_J)$ and matrix elements

$$\Theta_{mn}(t) = \sum_\ell D_{m\ell} \exp\left(-\Omega_\ell t\right) D^{-1}_{\ell n}, \tag{40a}$$

$$J_{mn}(t) = \Pi_{mn} - \sum_{m',n'} \Pi_{m'n'} \Theta^*_{mm'}(t)\Theta_{nn'}(t), \tag{40b}$$

we verify, after a rather lengthy calculation, that the evolution of the initial network state (38) can be described either through the characteristic function

$$\chi(\{\eta_m\},t) = \sum_J p_J \int \mathrm{d}s_J \Lambda(s_J) \int \mathrm{d}r\, \Lambda^*(r_J)\langle\{\beta_m(r_J)\}|\{\beta_m(s_J)\}\rangle \times \exp\left\{\sum_m [\eta_m K^*_m(r_J;t) - \eta^*_m K_m(s_J;t)] - \frac{1}{2}\sum_{m,n} \eta_m J_{mn}(t)\eta^*_n\right\}, \tag{41}$$

either by the Glauber–Sudarshan P-function

$$P(\{\xi_m\},t) = \frac{(2/\pi)^N}{\det \mathbf{J}} \sum_J p_J \int \mathrm{d}s_J \Lambda(s_J) \int \mathrm{d}r_J \Lambda^*(r_J)\langle\{\beta_m(r_J)\}|\{\beta_m(s_J)\}\rangle \times \exp\left\{-2\sum_{m,n} J^{-1}_{mn}(t)[\xi_m - K_m(s_J;t)][\xi_n - K_n(r_J;t)]^*\right\}, \tag{42}$$

or even by the Wigner distribution function

$$W(\{\xi_m\},t) = \frac{(2/\pi)^N}{\det \tilde{\mathbf{J}}} \sum_J p_J \int \mathrm{d}s_J \Lambda(s_J) \int \mathrm{d}r_J \Lambda^*(r_J)\langle\{\beta_m(r_J)\}|\{\beta_m(s_J)\}\rangle \times \exp\left\{-2\sum_{m,n} \tilde{J}^{-1}_{mn}(t)[\xi_m - K_m(s_J;t)][\xi_n - K_n(r_J;t)]^*\right\}. \tag{43}$$

Note that the difference between the Glauber–Sudarshan P-function and the Wigner distribution comes from the time-dependent function associated with the width of the their Gaussian function. Consequently, the Wigner function can be obtained from the Glauber–Sudarshan P-function through the substitution $\mathbf{J} \to \tilde{\mathbf{J}} = \mathbf{J}+\mathbf{I}$. Whereas the Glauber–Sudarshan

P-function diverges when there is no diffusion process such that $\mathbf{J} = \mathbf{0}$ (with all the reservoirs at 0 K), the width of the Wigner function presents an additional term $\mathbf{I}$ inhibiting any singularity.

For the case of 0 K reservoirs [11], the density operator of the network, to be used below, is given by

$$\rho_S(t) = \sum_J p_J \int \mathrm{d}s_J \Lambda(s_J) \int \mathrm{d}r_J \Lambda^*(r_J) \frac{\langle\{\beta_m(r_J)\}|\{\beta_m(s_J)\}\rangle}{\langle\{K_m(r_J;t)\}|\{K_m(s_J;t)\}\rangle} |\{K_m(s_J;t)\}\rangle\langle\{K_m(r_J;t)\}|. \tag{44}$$

4.2. A mixed superposition of Fock states

We now assume the initial network state to be a mixture of superposition of Fock states $|\Phi(0)\rangle_J = \sum_{x_1,\ldots,x_N} C^{(J)}_{x_1,\ldots,x_N} |x_1,\ldots,x_N\rangle$, where the parameter x_m indicates the number of photons in the mth oscillator while the coefficient $C^{(J)}_{x_1,\ldots,x_N}$ represents the probability amplitude associated with each state $|x_1,\ldots,x_N\rangle \equiv |\{x_m\}\rangle$ composing the whole superposition. The initial density operator is thus given by

$$\rho_S(0) = \sum_J p_J \sum_{\{x_m\}} \sum_{\{y_m\}} \left(C^{(J)}_{\{y_m\}}\right)^* C^{(J)}_{\{x_m\}} |\{x_m\}\rangle \langle\{y_m\}|, \tag{45}$$

where p_J is the probability associated with the state $|\Phi(0)\rangle_J$. Since the Fock state $|x_m\rangle$, of the mth oscillator, can be expanded as a superposition of coherent states of the form

$$|x_m\rangle = \mathcal{N}_m \int_0^{2\pi} \mathrm{d}\theta_m \, \mathrm{e}^{-ix_m\theta_m} |\beta_m \, \mathrm{e}^{i\theta_m}\rangle, \tag{46}$$

it is easy to note that the initial state (45) can be obtained by equation (38), identifying

$$\Lambda(r_J) \to \Lambda\left(\theta_J\right) = \sum_{\{x_m\}} C^{(J)}_{\{x_m\}} \prod_m \mathcal{N}_m \, \mathrm{e}^{-ix_m\theta_m}, \tag{47a}$$

$$\int \mathrm{d}r_J \to \int_0^{2\pi} \mathrm{d}\theta_m, \tag{47b}$$

$$\beta_m(r_J) \to \beta_m \, \mathrm{e}^{i\theta_m}, \tag{47c}$$

such that we can use the results of the previous subsection to obtain the characteristic function, the Glauber–Sudarshan P-function and Wigner distribution for a mixed superposition of pure Fock states. Alternatively, such functions may be directly computed from the initial state (45). Their expressions are presented in appendix B, where the density operator for a mixed superposition of pure Fock states is also presented.

5. Time-evolved dispersion coefficients

To analyze the time-dependent dispersion mechanism, due to the constant diffusion coefficients associated with the finite temperature of the reservoirs, we start by computing the Wigner distribution associated with the normal-mode oscillators. As depicted in figure 2, these oscillators, described by the Hamiltonian $H = H_0 + V$ (equations (7*a*) and (7*b*)), do not interact with each other, but they do interact with all the reservoirs. We, thus, rewrite the Wigner distribution (43) in a new coordinate frame $\{\tilde{\xi}_m\}$, obtained through the diagonalization of matrix $\tilde{\mathbf{J}}(t)$. This new framework follows from the rotation

$$\tilde{\boldsymbol{\xi}} = \xi \bullet \mathbf{U}(t); \qquad \tilde{\boldsymbol{\xi}}^* = \mathbf{U}^\dagger(t) \bullet \boldsymbol{\xi}^*, \tag{48}$$

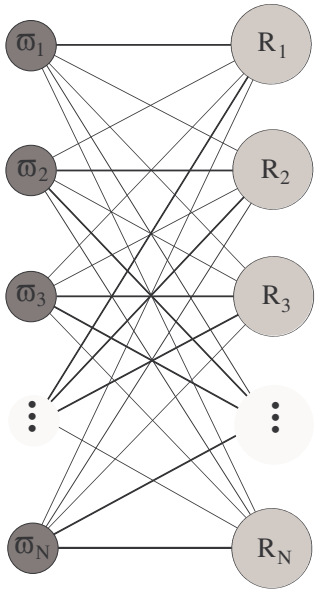

Figure 2. Sketch of a dissipative symmetric network of N noninteracting normal-mode oscillators, each one interacting with all the reservoirs.

where the unitary operation $\mathbf{U}(t)$ satisfies $\mathbf{U}^{\dagger}(t)\bullet\tilde{\mathbf{J}}(t)\bullet\mathbf{U}(t)=\mathfrak{D}(t)$. From this matrix relation, we obtain the evolved dispersion coefficients

$$\mathcal{D}_m(t)=\sum_{n,n'}U^{\dagger}_{mn}(t)\tilde{J}_{nn'}(t)U_{n'm}(t) \tag{49}$$

as the elements of the diagonal matrix $\mathfrak{D}(t)$. In this framework, the rotated Wigner distribution, written as

$$W(\{\tilde{\xi}_m\},t)=\sum_J p_J\int \mathrm{d}s_J\int \mathrm{d}r_J\, W(\{\tilde{\xi}_m\};r_J,s_J,t), \tag{50}$$

is composed by diagonal ($r_J=s_J$) and off-diagonal ($r_J\neq s_J$) elements defined by

$$\begin{aligned}W(\{\tilde{\xi}_m\};r_J,s_J,t)=&\frac{(2/\pi)^N}{\det\tilde{\mathbf{J}}}\Lambda^*(r_J)\Lambda(s_J)\langle\{\beta_m(r_J)\}|\{\beta_m(s_J)\}\rangle\\&\times\exp\left\{-\sum_m\frac{2}{\mathcal{D}_m(t)}[\tilde{\xi}_m-\tilde{K}_m(s_J;t)][\tilde{\xi}_m-\tilde{K}_m(r_J;t)]^*\right\},\end{aligned} \tag{51}$$

where $\tilde{\mathbf{K}}(r_J;t)=\mathbf{K}(r_J;t)\bullet\mathbf{U}(t)$ and $\tilde{\mathbf{K}}^*(r_J;t)=\mathbf{U}^{\dagger}(t)\bullet\mathbf{K}^*(r_J;t)$. The vector $\tilde{\mathbf{K}}(r_J;t)$ gives the excitation intensity of the mth normal-mode oscillator through $|\tilde{K}_m(r_J;t)|^2$. We stress that the larger or smaller values of $\mathcal{D}_m(t)$ depend on the network topology (contained within the matrix elements U_{mn}), apart from the regime of coupling strengths between the oscillators (contained within the matrix elements $\tilde{J}_{mn}(t)$). As a particular example of this dependence, we consider a degenerate symmetric network, i.e., a degenerate network of N oscillators, all of them interacting with each other, where $\{\omega_m\}=\omega$, $\{\lambda_{mn}\}=\lambda$, $\{\gamma_m\}=\gamma$, $\{\tilde{\gamma}_m\}=\tilde{\gamma}$ and $\{\bar{n}_m\}=\bar{n}$. In this case, in the strong coupling regime, we obtain the expression

$$\tilde{J}_{mn}(t)=\delta_{mn}+\frac{2\bar{n}}{N}(1-\mathrm{e}^{-N\gamma t}), \tag{52}$$

and the dispersion coefficients

$$\mathcal{D}_m(t) = \begin{cases} \tilde{J}_{mm}(t) - \tilde{J}_{mn}(t) & = & 1 & \text{for} & m \neq N, \\ \tilde{J}_{mm}(t) + (N-1)\,\tilde{J}_{mn}(t) & = & 1 + 2\bar{n}(1 - \mathrm{e}^{-N\gamma t}) & \text{for} & m = N, \end{cases} \tag{53}$$

showing that $\tilde{J}_{mn}(t) \neq 0$ can reduce or enhance the strength of the dispersion coefficients $\mathcal{D}_m(t)$ associated with the normal-mode oscillators.

It must be observed that, although we are here addressing the normal-mode oscillators (depicted in figure 2), which do not interact with each other, the (weak or strong) interoscillator coupling regime does affect the strengths of the interactions between these normal modes with all the reservoirs.

5.1. Directional and mean dispersion times

From the above time-dependent dispersion coefficients (53) we define the directional dispersion time

$$\frac{1}{\tau_{\mathrm{diff}}^{(m)}} = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D}_m(t)\bigg|_{t=0}, \tag{54}$$

displaying a tendency to a significant spread of the peak—common to all elements (the diagonal and off-diagonal) of the Wigner function—associated with the mth normal-mode oscillator. Since each normal-mode oscillator defines a direction in the coordinate frame $\{\tilde{\xi}_m\}$, we are naturally led to define the mean dispersion time, associated with all the dimensions of the space, as the average value

$$\frac{1}{\tau_{\mathrm{diff}}} = \frac{1}{N}\sum_m \frac{1}{\tau_{\mathrm{diff}}^{(m)}} = \frac{1}{N}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Tr}\,\mathfrak{D}(t)\bigg|_{t=0}. \tag{55}$$

The average dispersion time becomes useful to compute the decoherence time of any network state when complemented with the estimated time for a significant decay of the peaks associated with the interference terms of the Wigner function ($r_j \neq s_j$), to be defined below as τ_{int}.

As an illustrative example of the above theory, below we analyze the dispersion coefficients $\mathcal{D}_m(t)$ for the weak and strong coupling regimes considering the case of a degenerate symmetric network.

5.1.1. The weak coupling regime. In the weak coupling regime, the matrix $\tilde{\mathbf{J}}(t)$, already in a diagonal form, is defined by the elements $\tilde{J}_{mn}(t) = [1 + 2\bar{n}(1 - \mathrm{e}^{-\gamma t})]\delta_{mn}$, such that $\mathbf{U} = \mathbf{1}$. In this regime, all the dispersion coefficients are equal to

$$\mathcal{D}_m(t) = \mathcal{D}(t) = 1 + 2\bar{n}(1 - \mathrm{e}^{-\gamma t}). \tag{56}$$

The average dispersion time becomes

$$\tau_{\mathrm{diff}} = \frac{1}{2\bar{n}\gamma}, \tag{57}$$

showing, as expected, that the larger the temperature, the smaller the time required for a significant dispersion rate. In this case, the coefficients $\mathcal{D}_m(t)$ are mode independent and assume a common value, such that the spreads of the peaks associated with the diagonal terms of the Wigner function occur homogeneously in all directions.

5.1.2. The strong coupling regime. In the strong coupling regime, the elements of matrix $\tilde{\mathbf{J}}(t)$ are given by equation (52) and the dispersion coefficients by equation (53), showing that only the Nth normal-mode oscillator undergoes the dispersion process. For all the normal-mode oscillators but the Nth, the dispersion coefficients $\mathcal{D}_m(t)$, are counterbalanced by the dispersion rates $\tilde{J}_{mm}(t)$ and $\tilde{J}_{mn}(t)$ coming from the direct- and indirect-decay channels, respectively. The dispersion coefficients in this regime lead to the same mean dispersion time as that in equation (57), showing that the average dispersion effect comes entirely from the temperatures of the reservoirs. As to be demonstrated in the next section, this interesting result is not limited to the degenerate symmetric topology.

5.2. Dispersion and topology

Starting from equation (55) and noting that $\mathrm{Tr}\,\mathfrak{D}(t) = \mathrm{Tr}\,\tilde{\mathbf{J}}(t)$ ($\tilde{\mathbf{J}}(t) = \mathbf{J}(t) + \mathbf{I}$), with the elements of matrix $\mathbf{J}(t)$ given by equation (40*b*), we obtain the general expression

$$\tau_{\mathrm{diff}}^{-1} = \frac{2}{N}\mathrm{Tr}\,\boldsymbol{\Upsilon}, \tag{58}$$

applicable to whatever the network topology and the strength coupling regime between the oscillators, where

$$\mathrm{Tr}\boldsymbol{\Upsilon} = N\sum_{m,n}\tilde{\gamma}_m(\varpi_n)\bar{n}_m(\varpi_n)C_{nm}C_{mn}^{-1}. \tag{59}$$

We note that the information regarding the topology of the network is contained only in the product $C_{mn}^{-1}C_{nm}$ which acts as a normalized distribution function ($\sum_m C_{nm}C_{mn}^{-1} = 1$) when computing the average value of the dispersion rate given by equation (58).

We identify two general situations where, as in the case of a degenerate symmetric network, the dispersion mechanism becomes independent of the topology of the network. The first situation occurs (i) when identical reservoirs are assumed, such that $\tilde{\gamma}_m(\varpi_n)\bar{n}_m(\varpi_n) = \tilde{\gamma}(\varpi_n)\bar{n}(\varpi_n)$ and, consequently, $\mathrm{Tr}\,\boldsymbol{\Upsilon} = N\sum_n \tilde{\gamma}(\varpi_n)\bar{n}(\varpi_n)$, making the mean dispersion

$$\tau_{\mathrm{diff}}^{-1} = 2\sum_m \tilde{\gamma}(\varpi_m)\bar{n}(\varpi_m), \tag{60}$$

independent of the network topology. The second situation (ii) arises from the assumptions of Markovian white noise reservoirs and the low-temperature regime, where the normal-mode frequencies satisfy the relation $\hbar\varpi_m \gg k_BT$, with k_B being the Boltzmann constant. In this case we obtain $\tilde{\gamma}_m(\varpi_n)\bar{n}_m(\varpi_n) \approx \tilde{\gamma}_m\bar{n}_m$, such that $\mathrm{Tr}\,\boldsymbol{\Upsilon} = N\sum_m \tilde{\gamma}_m\bar{n}_m$, with $\bar{n}_m$ being computed around the average value of the normal-mode frequencies. The mean dispersion time, independent of the network topology, becomes

$$\tau_{\mathrm{diff}}^{-1} = 2\sum_m \tilde{\gamma}_m\bar{n}_m. \tag{61}$$

Both situations (i) and (ii) were considered in order to demonstrate that the mean dispersion time for both, weak and strong coupling regimes, is the same when considering a degenerate symmetric network. For any other situation, apart from (i) and (ii), the average dispersion rate becomes dependent on the network topology, apart from the reservoirs temperatures.

6. Collective decoherence rates

We start this section by noticing that, although we have derived the solution of the master equation when temperature effects are present (see appendix B), it is hard to analyze

decoherence using such an intricate solution. To circumvent this difficulty, it becomes appropriate to use the Wigner distribution function of the system, instead of the density operator, to estimate the decoherence time of a family of superposition states which are particular cases of the general state given by equation (38). This family of states is given by

$$|\psi_{1,\ldots,N}(0)\rangle = \mathcal{N}_{\pm}\left(\left|\underbrace{\alpha,\ldots,\alpha}_{R},\underbrace{-\alpha,\ldots,-\alpha}_{S},\underbrace{\beta,\ldots,\beta}_{N-R-S}\right\rangle \pm \left|\underbrace{-\alpha,\ldots,-\alpha}_{R},\underbrace{\alpha,\ldots,\alpha}_{S},\underbrace{\beta,\ldots,\beta}_{N-R-S}\right\rangle\right), \tag{62}$$

where R (S) indicates the number of oscillators in the coherent state α ($-\alpha$) in the first term of the superposition and $-\alpha$ (α) in the second term of the superposition. The remaining $N-R-S$ oscillators are in the coherent state β. We again stress that we consider a degenerate symmetric network where all the oscillators are indistinguishable. Therefore, by swapping the states of any two oscillators m and n, we obtain a state which is completely equivalent to equation (62). We also note that when $R = 1$ and $S = 0$, we obtain from (62) the superposition

$$|\tilde{\psi}_{1,\ldots,N}(0)\rangle = \mathcal{N}_{\pm}\left(|\alpha\rangle \pm |-\alpha\rangle\right)_1 \otimes |\{\beta_\ell\}\rangle\,, \tag{63}$$

where a 'Schrödinger cat'-like state is prepared in oscillator 1 while all the remaining oscillators are prepared in the coherent states β.

We start our calculation noting that for a pure two-level state $|\Psi\rangle = a\,|+\rangle + b\,|-\rangle$, whose density matrix is given by $\rho = a^*a\,|+\rangle\langle+| + b^*b\,|-\rangle\langle-| + a^*b\,|-\rangle\langle+| + ab^*\,|+\rangle\langle-|$, the ratio of the products between the diagonal and off-diagonal elements equals unity, i.e., $(a^*b)\,(ab^*)\,/\,(a^*a)\,(b^*b) = 1$. For an open system described by a mixed density matrix, however, this ratio decreases from unity. Bearing this in mind, we rewrite the Wigner function (50), to the discrete case where $\Lambda\left(r_j\right) = \sum_k \Lambda_k \delta\left(r_j - r_j^{(k)}\right)$, in a form

$$W(\{\tilde{\xi}_m\}, t) = \sum_{r,s=1}^{2} W_{r,s}(\{\tilde{\xi}_m\}, t), \tag{64}$$

with its diagonal ($r = s$) and off-diagonal ($r \neq s$) elements given by

$$W_{r,s}(\{\tilde{\xi}_m\}, t) = \frac{(2/\pi)^N}{\det \tilde{\mathbf{J}}} \Lambda_r^* \Lambda_s \left\langle \left\{\beta_m^r\right\} \middle| \left\{\beta_m^s\right\}\right\rangle \times \prod_m \exp\left\{-\frac{2}{\mathcal{D}_m(t)}\left[\tilde{\xi}_m - \tilde{K}_m^s(t)\right]\left[\tilde{\xi}_m - \tilde{K}_m^r(t)\right]^*\right\}, \tag{65}$$

where r and s (running from 1 to 2) label the product states composing the superposition (62).

Now, through the diagonal and off-diagonal elements of the Wigner function, we define the ratio

$$\Xi_{rs}(t) = \frac{W_{r,r}(\{\tilde{\xi}_m\}, t) W_{s,s}(\{\tilde{\xi}_m\}, t)}{W_{r,s}(\{\tilde{\xi}_m\}, t) W_{s,r}(\{\tilde{\xi}_m\}, t)} = \exp\left[\sum_m \left(\left|\beta_m^s - \beta_m^r\right|^2 - \frac{2}{\mathcal{D}_m(t)}\left|\sum_{m',n} U_{mm'}(t)\Theta_{m'n}(t)\left(\beta_n^r - \beta_n^s\right)\right|^2\right)\right]. \tag{66}$$

which turns out to be independent of the variables $\{\tilde{\xi}_m\}$ of the Wigner function, as desired. Moreover, for $t = 0$, such that $\Theta_{mn}(0) = \delta_{mn}$ and $\mathcal{D}_m(0) = 1$, we obtain $\Xi_{rs}(0) = \exp\left(-\sum_m \left|\beta_m^s - \beta_m^r\right|^2\right)$. In analogy with the above observation concerning the ratio

of the products between the diagonal and off-diagonal elements of a pure or mixed density matrix, the above defined ratio $\Xi_{rs}(t)$ offers a measure of the decoherence rate which follows from the function

$$\wp_{rs}(t) \equiv \frac{\Xi_{rs}(0)}{\Xi_{rs}(t)} = \exp\left[-2\sum_m \left(\left|\beta_m^s - \beta_m^r\right|^2 - \frac{1}{\mathcal{D}_m(t)}\left|\sum_{m',n} U_{mm'}(t)\Theta_{m'n}(t)\left(\beta_n^r - \beta_n^s\right)\right|^2\right)\right], \tag{67}$$

which equals unity for $t = 0$. The above deduction of the decay function (67) can also be developed for the general case of an initial continuous superposition state, instead of a discrete one.

6.1. The equivalence between the decays of the interference terms of both the Wigner function and the density operator: reservoirs at absolute zero

This subsection is devoted to demonstrating that the measure of the decoherence rate offered by equation (67) is equivalent to the one coming from the interference terms of the density operator, which is commonly used for the case of 0 K reservoirs. In fact, for reservoirs at 0 K, where $\mathcal{D}_m(t) = 1$, it is simple to verify that equation (67) reduces to

$$\wp_{rs}(t) = \exp\left[-2\sum_m \left(\left|\beta_m^s - \beta_m^r\right|^2 - \left|\sum_n \Theta_{mn}(t)\left(\beta_n^s - \beta_n^r\right)\right|^2\right)\right] = \left|\frac{\langle\{\beta_m^r\}|\{\beta_m^s\}\rangle}{\langle\{K_m^r(t)\}|\{K_m^s(t)\}\rangle}\right|^4, \tag{68}$$

where the coefficients $\langle\{\beta_m^r\}|\{\beta_m^s\}\rangle / \langle\{K_m^r(t)\}|\{K_m^s(t)\}\rangle$ are those coming from the interference terms of density operator (44), when considering a discrete case. Therefore, considering that decoherence times are usually estimated through the relation $\langle\{\beta_m^r\}|\{\beta_m^s\}\rangle / \langle\{K_m^r(\tau_D)\}|\{K_m^s(\tau_D)\}\rangle = \mathrm{e}^{-1}$, for the case of reservoirs at 0 K, we obtain from equation (68) the equivalent relation $\wp_{rs}(\tau_D) = \mathrm{e}^{-4}$, which gives the estimative of the decoherence time through the Wigner function.

6.2. Decay time of the interference terms

Now we are able to define the time τ_{int} for a significant decay of the peaks associated with the interference terms of the Wigner function ($r_j \neq s_j$). This is done, by generalizing the relation $\wp_{rs}(\tau_D) = \mathrm{e}^{-4}$, for the case of reservoirs at finite temperatures, to the equality

$$\wp_{rs}(\tau_{\mathrm{int}}) = \exp\left[-4N / \sum_m \mathcal{D}_m(\tau_{\mathrm{int}})\right], \tag{69}$$

that corresponds to measuring the decay of the interference terms of the Wigner function by deducting their spreadings, common to all the diagonal and off-diagonal elements, as we can see in equation (65). In other words, it is similar to analyzing the decay of the interference terms in a frame where the diagonal terms are frozen.

6.3. Decoherence time

Finally, to define a decoherence time τ_D, which takes into account both the dispersion and decay of the interference terms, we must consider both the above defined times: the mean

dispersion time τ_{diff} and the decay time of the interference terms of the Wigner functions τ_{int}. We thus define the relation

$$\frac{1}{\tau_D} = \frac{1}{\tau_{\text{diff}}} + \frac{1}{\tau_{\text{int}}}, \tag{70}$$

where τ_{diff}^{-1} only becomes relevant for particular initial states whose interference terms of the Wigner function are null, as occur, for example in the case $N = 1$, to the coherent state $|\alpha\rangle$, or when the excitation of the components of a superposition state is significantly smaller than unity. This will become clear in the example to be analyzed below for the degenerate symmetric network. In the first case, it is well known that a coherent state remains as such, even under a dissipative process, when considering a reservoir at 0 K. However, when considering a reservoir at finite temperature, the decoherence time of a coherent state $|\alpha\rangle$ can be estimated through our defined equations (58) and (59).

6.3.1. The weak coupling regime. The Wigner function associated with the pure state (62) in the weak coupling regime is obtained from equation (65) with $\mathbf{U} = \mathbf{1}$ and $\mathcal{D}_m(t) = \mathcal{D}(t) = 1 + 2\bar{n}\left(1 - \mathrm{e}^{-\gamma t}\right)$. Our defined decay function (67) thus becomes

$$\wp_{rs}(t) \equiv \exp[-8\mathcal{D}^{-1}(t)|\alpha|^2(R+S)(1+2\bar{n})(1-\mathrm{e}^{-\gamma t})]. \tag{71}$$

We estimate the decoherence time τ_D of the family of states (62) through the equality $\wp_{rs}(\tau_{\text{int}}) = \exp\left[-4\mathcal{D}^{-1}(\tau_{\text{int}})\right]$. The obtained result for the decay time and so for the decoherence time reads

$$\tau_D \approx \tau_{\text{int}} = \frac{1}{2|\alpha|^2\gamma}\frac{1}{(R+S)(1+2\bar{n})}, \tag{72}$$

which recovers the results in [10] for 0 K reservoirs ($\bar{n} = 0$). In equation (72) we have disregarded the mean dispersion time $\tau_{\text{diff}}^{-1} = 2\bar{n}\gamma$ since we assumed that the excitation $(R+S)|\alpha|^2$ is significantly larger than unity. Note that in the case where $R = N$ ($S = 0$) or $S = N$ ($R = 0$), given the initial entangled state $|\hat{\psi}_{1,\ldots,N}(0)\rangle = \mathcal{N}_{\pm}(|\alpha,\ldots,\alpha,\rangle \pm |-\alpha,\ldots,-\alpha\rangle)$, the decoherence time decreases as the number of network oscillators increases.

For the case of the 'Schrödinger cat'-like state in equation (63), we obtain the result

$$\tau_D \approx \tau_{\text{int}} = \frac{1}{2|\alpha|^2\gamma}\frac{1}{(1+2\bar{n})}. \tag{73}$$

Summarizing, when the weak coupling regime is considered, the decoherence time decreases with increasing temperature.

6.3.2. The strong coupling regime. From the Wigner function associated with the state $\left|\hat{\psi}_{1,\ldots,N}(0)\right\rangle$, derived from equation (65) and using the coefficients (53), we obtain in the strong coupling regime

$$\wp_{rs}(t) = \exp\left[-8\mathcal{D}_N^{-1}(t)|\alpha|^2N^2(1+2\bar{n})(1-\mathrm{e}^{-\gamma N t})\big/N\right]. \tag{74}$$

The estimated decay time τ_{int} of the interference terms of the Wigner functions is established through the inequality $\wp_{rs}(\tau_{\text{int}}) = \exp\{-4N/[N-1+\mathcal{D}_N(\tau_{\text{int}})]\} \leqslant \exp\left\{-4\mathcal{D}_N^{-1}(\tau_{\text{int}})\right\}$, such that

$$\tau_{\text{int}} \geqslant \frac{1}{2|\alpha|^2\gamma}\frac{1}{N^2(1+2\bar{n})}, \tag{75}$$

showing that the interference terms of the Wigner distribution decay at a faster rate than in the weak coupling regime. For the 'Schrödinger cat'-like state, equation (63), we obtain exactly the result shown in equation (73).

We finally note that, considering only the usual decay of the interference terms, given by $\wp_{rs}(\tau_D) = \exp(-4)$, the estimation of the decoherence time leads to inconsistent results which present negative values apart from singularities. For example, for the 'Schrödinger cat'-like state in equation (63), in the particular case $N = 1$, we obtain

$$\tau_D \approx \frac{1}{2\gamma[|\alpha|^2(1+2\bar{n}) - \bar{n}]}, \tag{76}$$

which has singularities at $\bar{n} = |\alpha|^2/(1-2|\alpha|^2)$ and becomes negative for $|\alpha|^2(1+2\bar{n}) < \bar{n}$. Therefore, the procedure adopted in equation (70) to estimate the decoherence time by separating both effects of dispersion and decay of the Wigner function interference terms is in fact better than the cruder approach where only the interference effects present in the decay function (67) are considered. Another example refers to the decoherence of a coherent state $|\alpha\rangle$, where the result $\tau_D \approx 1/2\gamma\bar{n}$ computed through our technique accounts exactly for the dispersion effect, apart from the decay rate γ, as expected. The usual procedure fails to give such an account.

As mentioned above, the analysis of the emergence of DFSs with the reservoirs at finite temperature is addressed in another work [19], where both, the collective effects of damping and dispersion, are managed together with the network topology to build up desired DFSs.

7. Computing the entropy and the entanglement degree through the Wigner function

The computation of the density operator of the network for the case of reservoirs at finite temperatures becomes a difficult task for the majority of the initial network states. Therefore, similar to our procedure for the analysis of decoherence, we next compute the entropy of the network using the Wigner functions as given by

$$\begin{aligned} S &= 1 - \mathrm{Tr}\rho_S^2 = 1 - \pi^N \int_{-\infty}^{\infty} \mathrm{d}^2\{\xi_m\}\, W^2(\{\xi_m\}, t) \\ &= 1 - \pi^N \int_{-\infty}^{\infty} \mathrm{d}^2\{\tilde{\xi}_m\} W^2(\{\tilde{\xi}_m\}, t), \end{aligned} \tag{77}$$

where the factor π^N was introduced to produce a null lower bound for the entropy. Using the integral result

$$\frac{1}{\pi}\int \mathrm{d}^2\eta_m \exp\left(a_m\eta_m^* - b_m\eta_m - c_m\eta_m^*\eta_m\right) = \frac{1}{c_m}\exp\left(-\frac{a_m b_m}{c_m}\right), \tag{78}$$

and the Wigner function given by equation (43), or equation (51), we obtain the general expression

$$\begin{aligned} S(t) = 1 - \int \mathrm{d}r \int \mathrm{d}r' \int \mathrm{d}s \int \mathrm{d}s' \Lambda^*(r)\Lambda^*(r')\Lambda(s)\Lambda(s')\langle\{\beta_m(r)\}|\{\beta_m(s)\}\rangle \\ \times\langle\{\beta_m(r')\}|\{\beta_m(s')\}\rangle \mathcal{P}_{rs,r's'}(t), \end{aligned} \tag{79}$$

which is applicable to any initial network state, where

$$\begin{aligned} \mathcal{P}_{rs,r's'}(t) = \frac{1}{\det \mathbf{J}} \exp\left\{-\sum_m [\upsilon_m(s,s')\upsilon_m^*(r,r') - \frac{1}{\mathfrak{D}_m(t)} \right. \\ \left. \times \left(\sum_{\ell,n} U_{\ell m}(t)\Theta_{\ell n}(t)\upsilon_n(s,s')\right)\left(\sum_{\ell,n} U_{\ell m}(t)\Theta_{\ell n}(t)\upsilon_n(r,r')\right)^*\right]\right\}, \end{aligned} \tag{80}$$

and $\upsilon_m(r,s) = \beta_m(r) - \beta_m(s)$. For the case where dissipation is absent, i.e., $\gamma_m(\omega) = \tilde{\gamma}_m(\omega) = 0$, we verify that $\mathcal{P}_{rs,r's'}(t) = 1$ and, consequently, $S = 0$. Conversely, when

$\gamma_m(\omega) \neq 0$ and $\tilde{\gamma}_m(\omega) \neq 0$, the purity loss follows from the decay of $\mathcal{P}_{rs,r's'}(t)$ which reduces to the function $\wp_{rs}(t)$,equation (67), that enters in the calculation of the decoherence time, under the conditions $\det \mathbf{J} = 1$ ($T = 0$ K), $r = r'$ and $s = s'$. As expected, the purity loss mechanism is intimately related to the decoherence one.

Focusing on the case when $\gamma_m(\omega) = \tilde{\gamma}_m(\omega) = 0$, the entanglement degree of a bipartite system described by a *pure* density operator ρ_{AB}—A and B standing for two complementary sets of network oscillators— can be computed through the reduced entropy (concurrence)

$$\mathcal{C} = 1 - \mathrm{Tr}_A \left[\mathrm{Tr}_B \rho_{AB}\right]^2 = 1 - \mathrm{Tr}_B \left[\mathrm{Tr}_A \rho_{AB}\right]^2, \tag{81}$$

which is given, through the joint Wigner function $W(\{\xi_A\}, \{\xi_B\}, t)$, as

$$\begin{aligned} \mathcal{C} &= 1 - \pi^{N_A} \int_{-\infty}^{\infty} \mathrm{d}^2 \{\xi_A\} \left[\int_{-\infty}^{\infty} \mathrm{d}^2 \xi_B W(\{\xi_A\}, \{\xi_B\}, t)\right]^2 \\ &= 1 - \pi^{N_B} \int_{-\infty}^{\infty} \mathrm{d}^2 \{\xi_B\} \left[\int_{-\infty}^{\infty} \mathrm{d}^2 \xi_A W(\{\xi_A\}, \{\xi_B\}, t)\right]^2, \end{aligned} \tag{82}$$

where N_A and N_B refer to the numbers of oscillators composing the sets A and B, respectively.

When the subsystems A and B are uncorrelated, such that $\rho_{AB} = \rho_A \otimes \rho_B$, the Wigner function is factorized as

$$W(\{\xi_A\}, \{\xi_B\}, t) = W(\{\xi_A\}, t)\, W(\{\xi_B\}, t), \tag{83}$$

and, consequently

$$\mathcal{C} = 1 - \pi^{N_B} \int_{-\infty}^{\infty} \mathrm{d}^2 \{\xi_B\}\, W^2(\{\xi_B\}, t) = 0, \tag{84}$$

as expected.

8. Concluding remarks

In the present work we have analyzed the effects of temperature in a network of dissipative quantum harmonic oscillators. Starting from a previous work where a general treatment of such a bosonic dissipative network was presented [11], in the case of reservoirs at 0 K, here we considered reservoirs at finite temperatures. Through the solution obtained for the normal-ordered characteristic function, we did compute formal expressions for the Glauber–Sudarshan P-function, the Wigner distribution function and the density operator for whatever the initial network state. An important point to be stressed is the relevance played by the Wigner function in the present context where the reservoirs are at finite temperature. In fact, it becomes hard to identify the main features associated with the dynamic of the network states through the density operator which appears to be an intricate expression. Through the Wigner function, however, the dispersion coefficients of the normal-mode oscillators are clearly identified as well as the decay of its interference terms. We also showed how to compute the entropy and the entanglement degree through the Wigner function. We thus emphasize that, despite the solutions we have derived for the master equations, we have used the Wigner distribution function to estimate the decoherence time of a family of superposition states of the network. Being easier to handle mathematically, the Wigner function circumvents the difficulties inherent to the involved solutions we have derived for the master equations.

We demonstrated that the dispersion coefficients $\mathcal{D}_m(t)$ associated with the normal-mode oscillators present completely different behaviors in both weak and strong coupling regimes. In the former case, where the indirect-decay channels do not take place, the dispersion coefficients are entirely related to the dissipative processes of the oscillators to their own reservoirs. In

this case the collective damping and dispersion effects are dismissible. However, in the later case, the dispersion coefficients $\mathcal{D}_m(t)$ are counterbalanced by the dispersion rates $\tilde{J}_{mm}(t)$ and $\tilde{J}_{mn}(t)$ coming from the direct- and indirect-decay channels, respectively. In this case, the collective damping and dispersion effects emerge from the fact that all network oscillators interact with all the reservoirs due to the strong coupling between each other. In fact, in the strong coupling regime, the individual oscillators cannot account for the dynamic of the whole network, which must be described through the collective normal-mode oscillators. Differently, in the weak coupling regime, the network dynamic follows directly from those of the individual oscillators.

In summary, we have presented an analysis of the mechanisms for handling the dispersion coefficients $\mathcal{D}_m(t)$ in the strong coupling regime, by manipulating the dispersion rates $\tilde{J}_{mm}(t)$ and $\tilde{J}_{mn}(t)$ through the nature and the temperature of the reservoirs, apart from the network topology. Such an approach was explored in [19] to demonstrate the possibility of the emergence of DFSs in a network of dissipative oscillators even with the reservoirs at finite temperatures.

We have also presented a technique to estimate the decoherence time of network states which separates the effects of dispersion from the decay of the interference terms in the Wigner distribution function. Our technique overcomes the difficulties that show up with negative values and singularities arising from the usual definition of the decoherence time based only on the decay of interference terms. We have computed the decoherence time for some particular states of the network, leaving for future work [19] the analysis of the emergence of DFSs under temperature effects.

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Appendix A. Matrix equation

The solution of an arbitrary matrix equation of the form $\mathbf{M} \bullet \mathbf{X} + \mathbf{X} \bullet \mathbf{N}^{\top} = \mathbf{P}$ (for an unknown $\mathbf{X}$) can be obtained through the solution of the system

$$[(\mathbf{I} \otimes \mathbf{M}) + (\mathbf{N} \otimes \mathbf{I})] \bullet \mathrm{vec}(\mathbf{X}) \equiv \mathrm{vec}(\mathbf{P}), \tag{A.1}$$

following from the inverse of $[(\mathbf{I} \otimes \mathbf{M}) + (\mathbf{N} \otimes \mathbf{I})]$, given by

$$\mathrm{vec}(\mathbf{X}) = [(\mathbf{I} \otimes \mathbf{M}) + (\mathbf{N} \otimes \mathbf{I})]^{-1} \bullet \mathrm{vec}(\mathbf{P}), \tag{A.2}$$

where the notation vec $(\mathbf{P})$ was defined in equation (32). Before computing the elements of the inverse matrix $(\mathbf{I} \otimes \mathbf{M}) + (\mathbf{N} \otimes \mathbf{I}) \equiv \mathbf{Q}$, it is useful to observe some important properties of $\mathbf{Q}$:

(i) The eigenvalues of matrix $\mathbf{Q}$, defined by ε_{ij}, are obtained through the direct sum of the eigenvalues ϵ_i and $\tilde{\epsilon}_i$ of matrices $\mathbf{M}$ and $\mathbf{N}$, such that

$$\varepsilon_{ij} = \tilde{\epsilon}_i + \epsilon_j. \tag{A.3}$$

(ii) The eigenvectors of matrix $\mathbf{Q}$ are obtained through the tensor product

$$\vartheta_{ij} = \tilde{\nu}^{(i)} \otimes \nu^{(j)}, \tag{A.4}$$

where $\nu^{(i)}$ and $\tilde{\nu}^{(i)}$ describe the eigenvector associated with the eigenvalues ϵ_i and $\tilde{\epsilon}_i$. In fact, knowing the eigenvalues and eigenvectors of matrices $\mathbf{M}$ and $\mathbf{N}$, we can easily verify that ϑ_{ij}

defines the desired eigenvector, associated with the eigenvalue ε_{ij}, since

$$\begin{aligned}[(\mathbf{I}\otimes\mathbf{M})+(\mathbf{N}\otimes\mathbf{I})]\bullet(\tilde{\nu}^{(i)}\otimes\nu^{(j)}) &= \tilde{\nu}^{(i)}\otimes(\mathbf{M}\bullet\nu^{(j)})+(\mathbf{N}\bullet\tilde{\nu}^{(i)})\otimes\nu^{(j)}\\ &= (\tilde{\epsilon}_i+\epsilon_j)(\tilde{\nu}^{(i)}\otimes\nu^{(j)})\\ &= \varepsilon_{ij}(\tilde{\nu}^{(i)}\otimes\nu^{(j)}).\end{aligned} \tag{A.5}$$

Appendix B. Alternative expression for the evolution of a mixed superposition of Fock states

We verify that the evolution of the initial state (45) can be characterized, using the same definitions (40*a*) and (40*b*), through the characteristic function

$$\begin{aligned}\chi(\{\eta_m\},t) = &\sum_J\sum_{\{x_m\}}\sum_{\{y_m\}}p_J\left(C^{(J)}_{\{y_m\}}\right)^*C^{(J)}_{\{x_m\}}\exp\left(-\frac{1}{2}\sum_{m,n}\eta_m J_{mn}(t)\eta_n^*\right)\\ &\times\prod_\ell\left[\sum_{j_\ell=0}^{x_\ell}\frac{\sqrt{y_\ell!x_\ell!}}{j_\ell!\,(x_\ell-j_\ell)!\,(y_\ell-x_\ell+j_\ell)!}\right.\\ &\left.\times\left(\sum_m\eta_m\Theta^*_{m\ell}(t)\right)^{y_\ell-x_\ell+j_\ell}\left(-\sum_m\eta_m^*\Theta_{m\ell}(t)\right)^{j_\ell}\right],\end{aligned} \tag{B.1}$$

which leads to the Wigner distribution function given in terms of derivatives as

$$\begin{aligned}W(\{\xi_m\},t) = &\frac{(2/\pi)^N}{\det\tilde{\mathbf{J}}}\sum_J\sum_{\{x_m\}}\sum_{\{y_m\}}p_J\left(C^{(J)}_{\{y_m\}}\right)^*C^{(J)}_{\{x_m\}}\\ &\times\left(\prod_\ell\sum_{j_\ell=0}^{x_\ell}\frac{\sqrt{y_\ell!x_\ell!}}{j_\ell!\,(x_\ell-j_\ell)!\,(y_\ell-x_\ell+j_\ell)!}\lim_{\varepsilon_\ell\to0}\frac{\partial^{y_\ell-x_\ell+2j_\ell}}{\partial\,(\varepsilon_\ell)^{j_\ell}\,\partial\left(\varepsilon_\ell^*\right)^{y_\ell-x_\ell+j_\ell}}\right)\\ &\times\exp\left[-2\sum_{m,n}\tilde{J}^{-1}_{mn}(t)\left(\xi_m-\sum_\ell\varepsilon_\ell\Theta_{m\ell}(t)\right)\left(\xi_n^*-\sum_\ell\varepsilon_\ell^*\Theta^*_{n\ell}(t)\right)\right].\end{aligned} \tag{B.2}$$

The above distribution can also be given explicitly in the form

$$\begin{aligned}W(\{\xi_m\},t) = &\frac{(2/\pi)^N}{\det\tilde{\mathbf{J}}}\sum_J\sum_{\{x_m\}}\sum_{\{y_m\}}p_J\left(C^{(J)}_{\{y_m\}}\right)^*C^{(J)}_{\{x_m\}}\left(\prod_\ell\sum_{q_\ell=0}^{x_\ell}\frac{\sqrt{y_\ell!x_\ell!}}{(x_\ell-q_\ell)!}\right)\\ &\times\left[\prod_{\ell,\ell'}\sum_{R_{\ell\ell'}=0}^{R_{\ell,\ell'-1}}\Delta_{\ell'\ell}(\{R_{n,n'}\};t)F\left(y_{\ell'}-x_{\ell'}+q_{\ell'}\right.\right.\\ &\left.\left.-\sum_{i=1}^{\ell-1}\left(R_{i,\ell'-1}-R_{i,\ell'}\right)-\left(R_{\ell,\ell'-1}-R_{\ell,\ell'}\right)\right)\right]\\ &\times\left(\prod_\ell\Lambda_\ell\left(\left\{R_{n,n'}\right\},\left\{\xi_p\right\};t\right)\frac{\left[2\sum_{m,n}\tilde{J}^{-1}_{mn}(t)\Theta_{m\ell}(t)\xi_n^*\right]^{R_{\ell N}}}{R_{\ell N}!}\right)\\ &\times\exp\left(-2\sum_{m,n}\xi_m\tilde{J}^{-1}_{mn}\xi_n^*\right),\end{aligned} \tag{B.3}$$

where we have defined

$$R_{\ell,0} = q_\ell, \tag{B.4a}$$

$$\Delta_{mn}(\{R_{\ell,\ell'}\}; t) = \frac{1}{\left(R_{n,m-1} - R_{n,m}\right)!}\left[-2\sum_{\ell,\ell'}\tilde{J}^{-1}_{\ell\ell'}(t)\Theta_{\ell n}(t)\Theta^*_{\ell' m}(t)\right]^{R_{n,m-1}-R_{n,m}}, \tag{B.4b}$$

$$\Lambda_m(\{R_{\ell,\ell'}\}, \{\xi_p\}; t) = \frac{\left[2\sum_{\ell,\ell'}\tilde{J}^{-1}_{\ell\ell'}(t)\xi_\ell\Theta^*_{\ell' m}(t)\right]^{y_m - x_m + q_m - \sum_\ell\left(R_{\ell,m-1}-R_{\ell,m}\right)}}{\left[y_m - x_m + q_m - \sum_\ell\left(R_{\ell,m-1} - R_{\ell,m}\right)\right]!}, \tag{B.4c}$$

with

$$F(x) = \begin{cases} 1 & \text{for} \quad x \geqslant 0 \\ 0 & \text{for} \quad x < 0 \end{cases}. \tag{B.5}$$

As noted in section 4, we remember that the Glauber–Sudarshan P-function $P(\{\xi_m\}, t)$ can be derived from the Wigner distribution by replacing $\tilde{J}_{mn}$ by J_{mn}. Using such a P-function we obtain a compact expression of the evolved density operator associated with the initial state (45), given by

$$\begin{aligned}\rho_S(t) = \sum_J \sum_{\{x_m\}}\sum_{\{y_m\}} p_J \left(C^{(J)}_{\{y_m\}}\right)^* C^{(J)}_{\{x_m\}} \\ \times \left\{\prod_n \sum_{q_n=0}^{x_n} \sum_{i_n, j_n, k_n} \frac{(-1)^{k_n}}{k_n!\sqrt{i_n! j_n!}} \frac{\sqrt{x_n! y_n!}}{q_n!\,(x_n - q_n)!\,(y_n - x_n + q_n)!}\right. \\ \times \lim_{\varepsilon_n \to 0} \frac{\partial^{i_n + j_n + 2k_n}}{\partial\,(\varepsilon_n)^{j_n + k_n}\,\partial\left(\varepsilon^*_n\right)^{i_n + k_n}} \left[\left(\sum_m \varepsilon_m \Theta^*_{mn}(t)\right)^{y_n - x_n + q_n}\right. \\ \left.\left. \times \left(\sum_m \varepsilon^*_m \Theta_{mn}(t)\right)^{q_n}\right] |i_n\rangle\langle j_n|\right\} \exp\left(\frac{1}{2}\sum_{m,n} \varepsilon_m J_{mn}(t) \varepsilon^*_n\right).\end{aligned} \tag{B.6}$$

By defining the parameters

$$R_{\ell 0} = j_\ell + k_\ell, \tag{B.7a}$$

$$S_{\ell 0} = i_\ell + k_\ell, \tag{B.7b}$$

$$S_{\ell N} = 0, \tag{B.7c}$$

$$\delta(x) = \begin{cases} 1 & \text{if} \quad x = 0 \\ 0 & \text{if} \quad x \neq 0 \end{cases}, \tag{B.7d}$$

we, alternatively, obtain the explicit form of the density operator

$$\begin{aligned}\rho_S(t) = \sum_J \sum_{\{x_m\}}\sum_{\{y_m\}} p_J \left(C^{(J)}_{\{y_m\}}\right)^* C^{(J)}_{\{x_m\}} \left\{\prod_\ell \sum_{q_\ell=0}^{x_\ell} \sum_{i_\ell, j_\ell, k_\ell=0}^{\infty} \frac{\sqrt{y_\ell! x_\ell!}}{(x_\ell - q_\ell)!} \frac{(-1)^{k_\ell}\,(i_\ell + k_\ell)!\,(j_\ell + k_\ell)!}{k_\ell!\sqrt{i_\ell! j_\ell!}}\right. \\ \times \left[\prod_{\ell'} \sum_{S_{\ell,\ell'}=0}^{S_{\ell,\ell'-1}} \sum_{K_{\ell,\ell'}=0}^{S_{\ell,\ell'-1}-S_{\ell\ell'}} \sum_{R_{\ell,\ell'}=0}^{R_{\ell,\ell'-1}} \left(\frac{1}{2} J_{\ell'\ell}(t)\right)^{S_{\ell,\ell'-1}-S_{\ell,\ell'}-K_{\ell,\ell'}}\right. \\ \left.\left. \times \frac{[\Theta_{\ell\ell'}(t)]^{K_{\ell,\ell'}}\left[\Theta^*_{\ell\ell'}(t)\right]^{R_{\ell,\ell'-1}-R_{\ell,\ell'}}}{\left(R_{\ell,\ell'-1} - R_{\ell,\ell'}\right)!\left(S_{\ell,\ell'-1} - S_{\ell,\ell'} - K_{\ell,\ell'}\right)! K_{\ell,\ell'}!}\right] \delta\left(q_\ell - \sum_{\ell'} K_{\ell',\ell}\right)\end{aligned} \tag{B.8}$$

$$\times \delta\left[R_{\ell,N} - \sum_{\ell'}\left(S_{\ell',\ell-1} - S_{\ell',\ell} - K_{\ell',\ell}\right)\right]\delta$$
$$\times\left[y_\ell - x_\ell + q_\ell - \sum_{\ell'}\left(R_{\ell',\ell-1} - R_{\ell',\ell}\right)\right]|i_\ell\rangle\langle j_\ell|\Bigg\}. \tag{B.8}$$

For the case where all the reservoirs are at 0 K, so that $J_{mn} = 0$, only the terms with $K_{\ell,\ell'} = S_{\ell,\ell'-1} - S_{\ell,\ell'}$ survive in the summation over $K_{\ell,\ell'}$ of expression (B.8). Therefore, at 0 K, the density operator (B.8) reduces to the expression

$$\rho_S(t) = \sum_J \sum_{\{x_m\}}\sum_{\{y_m\}} p_J \left(C^{(J)}_{\{y_m\}}\right)^* C^{(J)}_{\{x_m\}} \left(\prod_\ell \sum_{q_\ell=0}^{x_\ell} \frac{\sqrt{y_\ell! x_\ell!}}{(x_\ell - q_\ell)!} \sum_{k_\ell=0}^{\infty} \frac{(-1)^{k_\ell}}{k_\ell!}\right)$$
$$\times |\mathcal{F}(\{q_\ell\},\{k_\ell\},t)\rangle\langle\mathcal{F}(\{y_\ell - x_\ell + q_\ell\},\{k_\ell\},t)| \tag{B.9}$$

as already presented in [11], where we have defined, with $S_{\ell N} = 0$, the superposition of product states

$$|\mathcal{F}(\{q_\ell\},\{k_\ell\},t)\rangle = \bigotimes_\ell \sum_{j_\ell=0}^{\infty} \frac{(j_\ell + k_\ell)!}{\sqrt{j_\ell!}} \left(\prod_{\ell'} \sum_{S_{\ell,\ell'}=0}^{S_{\ell,\ell'-1}} \frac{[\Theta_{\ell\ell'}(t)]^{S_{\ell,\ell'-1} - S_{\ell,\ell'}}}{\left(S_{\ell,\ell'-1} - S_{\ell,\ell'}\right)!}\right)$$
$$\times\delta\left(q_\ell - \sum_{\ell'}\left(S_{\ell',\ell-1} - S_{\ell',\ell}\right)\right)|j_\ell\rangle. \tag{B.10}$$

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